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ADDENDUM

The symmetrized Fermi function and its transforms

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Abstract. The method given in our paper is extended to odd functions, and the discussion is clarified.

1. Introduction

Our paper [1] concerned a simple method for evaluating integral transforms of a function $h(x)$ of the form

$$I = \int_0^\infty h(x) F(x-y)\,\mathrm{d}x \tag{1}$$

where

$$F(x-y) = \frac{1}{1+\mathrm{e}^{x-y}} \tag{2}$$

is the usual Fermi function in dimensionless units. In the case that $h(x) = h(-x)$ is *even*, it is convenient to introduce the symmetrized Fermi function

$$\rho_S(x,y) = \frac{1}{1+\mathrm{e}^{x-y}} + \frac{1}{1+\mathrm{e}^{-x-y}} - 1 = \frac{1}{1+\mathrm{e}^{x-y}} - \frac{1}{1+\mathrm{e}^{x+y}} = \frac{\sinh y}{\cosh x + \cosh y} \tag{3}$$

which is clearly an even function of x. We then showed that I_S, the transform with ρ_S, is given by the Blankenbecler formula [2]:

$$I_S = I - J = \frac{\pi D}{\sin \pi D} H(y) \tag{4}$$

where

$$H(x) = \int_0^x h(x')\,\mathrm{d}x' \tag{5}$$

and J is the integral with the function $F(x+y)$ from the second term in the next to last expression of equation (3):

$$J = \int_0^\infty h(x) F(x+y)\,\mathrm{d}x. \tag{6}$$

We now realize that the case of *odd* functions $h(x) = -h(-x)$ can be handled by the simple device of adding rather than subtracting the two pieces of ρ_S. Consider

$$\rho_C(x,y) = \frac{1}{1+\mathrm{e}^{x-y}} + \frac{1}{1+\mathrm{e}^{x+y}} = \frac{\mathrm{e}^{-x} + \cosh y}{\cosh x + \cosh y}. \tag{7}$$

One can then follow the previous derivation to show that the transform with ρ_C is

$$I_C = I + J = \frac{\pi D}{\sin \pi D} H(y) \tag{8}$$

which is the same result as before.

In many cases one really wants I, the transform with the usual Fermi function. Then the 'error term' J has to be added to I_S or subtracted from I_C. Both cases can be combined as I_S if we simply redefine J as

$$J = \int_0^{\infty} h(-x) F(x+y)\, \mathrm{d}x \tag{9}$$

so that the parity of $h(x)$ sets the proper sign. Further, this prescription works even if $h(x)$ does not have definite parity, because we can suppose that it has been divided into its even and odd parts, and each one transformed accordingly. If $h(x)$ is not defined for negative x, one may still be able to extend the definition in an appropriate way. For example, if there is a power series expansion or Fourier expansion for small x, that can be used to make the extension.

We note that $\rho_C(x=0, y) = 1$, $\rho_C(x=y, y) = 0.5(1 + \mathrm{e}^{-y}/\cosh(y))$, so for large enough y one can still consider y to be the half-density radius, just as for $\rho_S(x, y)$. While $\rho_C(x, y)$ has no particular symmetry as a function of x,

$$\rho_C(x, y) - 1 = \frac{-\sinh x}{\cosh x + \cosh y} \tag{10}$$

is an odd function of x. However this symmetry is not really used in the derivation of the Blankenbecler formula.

In effect, we already used this extended result in our paper [1] in evaluating the transforms of Bessel functions. However, what was incorrect was the statement concerning the error in using the Blankenbecler formula equation (8). If the aim is to have the transform with the usual Fermi function, as in equation (1), then the error term J simply has the opposite sign for odd functions. On the other hand, in nuclear physics applications, one may prefer to use the symmetrized Fermi function $\rho_S(x, y)$ as a model. In that case the error would be twice J for odd functions $h(x)$.

In most applications, the integral J gives a small correction to the result given by I_S. Then the Blankenbecler formula is a convenient way to obtain the desired transform.

2. A simple example

Consider the integral

$$\Omega_p \equiv \int_0^{\infty} \frac{x^p\, \mathrm{d}x}{1 + \mathrm{e}^x}. \tag{11}$$

This is a special case of equations (1), (2) with $y = 0$ and $h(x) = x^p$. The exact result is

$$\Omega_p = \int_0^{\infty} x^p \mathrm{e}^{-x}\, \mathrm{d}x (1 - \mathrm{e}^{-x} + \cdots + (-)^n \mathrm{e}^{-nx} + \cdots) = \sum_{n=0} (-)^n \int_0^{\infty} x^p \mathrm{e}^{-(n+1)x}\, \mathrm{d}x$$

$$= \sum_{n=0} (-)^n \frac{\Gamma(p+1)}{(n+1)^{p+1}} = p!\,\eta(p+1) \tag{12}$$

where we follow Abramowitz [3] in defining

$$\eta(n) = (1 - 2^{1-n})\zeta(n) \tag{13}$$

as a variant of Riemann's zeta-function.

To use Blankenbecler's formula equation (4) plus J, equation (9), we choose $H(x) = x^{p+1}/(p+1)$. One sees immediately that at $y = 0$ all the derivatives of $H(y)$ vanish, except $H^{(p+1)}(0) = p!$. In the case of p *odd*, from (A.1) of [1] we have

$$I_S = 2p!\eta(p+1) \tag{14}$$

using the fact that the c_ν defined in equation (A.2) of our article are just the functions $\eta(\nu)$. Thus I_S turns out to be *twice* the exact result. If instead p is *even*, then all the applicable (even) derivatives of $H(y)$ at $y = 0$ vanish and one has $I_S = 0$.

The evaluation of J is straightforward. From equation (9)

$$J = (-)^p \sum_{k=1} (-)^{k+1} \int_0^\infty \mathrm{e}^{-kx} x^p \, \mathrm{d}p = (-)^p \sum_{k=1} (-)^{k+1} k^{-(p+1)} p! = (-)^p p! \eta(p+1). \tag{15}$$

So in either case, we recover the exact result. Of course, by setting $y = 0$, we have ensured that the Fermi function is far from being a step function, and this has maximized the role of the 'error term'.

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